

# On algebraic congruences

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**Abstract** In this paper, by studying the structure of congruences on commutative semirings, the system of polynomial (congruence) equations over semirings and their zeros are considered. In particular, for commutative semirings  $A \subset B$  and a congruence  $\rho$  on  $B$ , if  $\rho$  is a prime congruence, then the set of zeros of the system of polynomial  $\rho$ -equations in an affine space  $B^n$ , which are called the  $\rho$ -algebraic varieties here, is shown to satisfy the axiom of closed sets, and forms a (Zariski) topology on  $B^n$ . Some results of their structures are obtained.

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## 1. Introduction

As known, ideals are very important in studying ring structure and the polynomial equations over rings, and, in a commutative ring, the ideal and the congruence corresponds to each other equivalently. Yet, in a commutative semiring, this changes, and congruences seem to behave more natural than ideals.

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Our motivation here is to consider the problem of solving polynomial equations over semirings, like the same important problem in algebraic geometry on rings. In fact, by studying the structure of congruences on commutative semirings, we consider the system of polynomial (congruence) equations over semirings and their zeros. In particular, for commutative semirings  $A \subset B$  and a congruence  $\rho$  on  $B$ , we show that, if  $\rho$  is a prime congruence, then the set of zeros of the system of polynomial  $\rho$ -equations in an affine space  $B^n$ , which are called the  $\rho$ -algebraic varieties here, satisfies the axiom of closed sets, and forms a (Zariski) topology on  $B^n$ . Some results of their structures are obtained. These facts show one possible way among others to consider the problem of solving polynomial (congruence) equations in semirings comparing with the same problem of algebraic geometry on rings.

The contents of this paper are arranged as follows.

In section 2, on a commutative semiring, we study prime congruence, maximal congruence, congruence generated by a subset and a few other types congruence, and obtain some results about their structures (see Propositions 2.6, 2.10, 2.12, 2.13, 2.16, 2.18, 2.19, 2.21, 2.23 and 2.25).

In section 3, we give a simple relation between ideals and congruences on a commutative semiring (see Prop.3.2). This also shows that such relation is different from the one in a commutative ring.

In section 4, we consider the problem of solving polynomial equations over semirings. for commutative semirings  $A \subset B$  and a congruence  $\rho$  on  $B$ , we show that, if  $\rho$  is a prime congruence, the set of the  $\rho$ -algebraic varieties in an affine

space  $B^n$ , forms a (Zariski) topology on  $B^n$  (see Theorem 4.4). Some results of their structure are obtained (see Theorems 4.10, 4.12; and Propositions 4.7, 4.9, 4.11).

## 2. The structure of congruences

Let  $(A, +, \cdot)$  be a semiring, that is,  $A$  is a non-empty set on which we have defined operations of addition and multiplication satisfying the following four conditions (see [G, p.1]):

- (1)  $(A, +)$  is a commutative monoid with identity element 0;
- (2)  $(A, \cdot)$  is a monoid with identity element  $1 \neq 0$ ;
- (3)  $a(b + c) = ab + ac$  and  $(a + b)c = ac + bc$  ( $\forall a, b, c \in A$ );
- (4)  $0a = 0 = a0$  ( $\forall a \in A$ ).

The semiring  $A$  is commutative if the monoid  $(A, \cdot)$  is commutative. In the following, unless otherwise stated, the semiring always means a commutative semiring.

Now for a commutative semiring  $A$ , recall that a (binary) relation on  $A$  is a subset of  $A \times A$  (see [Ho, p.14]), and a relation  $\rho$  of  $A$  is a congruence if it is an equivalence relation satisfying the following condition (see [G, p.4]):

$$(a, b), (c, d) \in \rho \implies (a + c, b + d) \in \rho \text{ and } (ac, bd) \in \rho \text{ } (a, b, c, d \in A).$$

So an equivalence relation  $\rho$  is a congruence on  $A$  if and only if  $\rho$  is a subsemiring of  $A \times A$ . We denote the identity congruence by  $\text{id}_A$ , i.e.,  $\text{id}_A = \{(a, a) : a \in A\}$ .

For a congruence  $\rho$  of  $A$ , it is easy to see that  $\rho = A \times A$  if and only if  $(1, 0) \in \rho$ . A congruence  $\rho \neq A \times A$  is called a proper congruence. We denote by  $\overline{A} = A/\rho$  the quotient semiring of  $A$  by a congruence  $\rho$ , and let  $\overline{c} = c \bmod \rho \in \overline{A}$  for any  $c \in A$ .

Moreover, for any positive integer  $n$  and subset  $C$  of  $A^n$ , where  $A^n = \{(a_1, \dots, a_n) :$

$a_1, \dots, a_n \in A\}$ , we denote  $C/\rho = \{(\overline{c_1}, \dots, \overline{c_n}) : (c_1, \dots, c_n) \in C\} \subset (A/\rho)^n$ . For  $0 \neq a \in A$ , if there exists an element  $0 \neq b \in A$  such that  $a \cdot b = 0$ , then  $a$  is called a zero divisor of  $A$ . If  $A$  contains no zero divisors, then  $A$  is called a semidomain. If  $A$  is a semidomain, and  $(A \setminus \{0\}, \cdot)$  is a multiplicative group, then  $A$  is called a semifield. For example,  $(\mathbb{Z}_{\geq 0}, +, \cdot)$  is a semidomain, and  $(\mathbb{Q}_{\geq 0}, +, \cdot)$  is a semifield, where  $\mathbb{Z}_{\geq 0}$  (resp.  $\mathbb{Q}_{\geq 0}$ ) is the set of non-negative integers (resp. rational numbers).

**Definition 2.1.** Let  $A$  be a commutative semiring, and  $\rho$  be a congruence of  $A$  such that  $\rho \neq A \times A$ .

(1) If  $\rho$  satisfies the following condition:

for  $(a, b), (c, d) \in A \times A$ ,  $(ac + bd, ad + bc) \in \rho \Rightarrow (a, b) \in \rho$  or  $(c, d) \in \rho$ .

Then  $\rho$  is called a prime congruence.

(2) If  $\rho$  satisfies the following condition:

$(ab, 0) \in \rho \Rightarrow (a, 0) \in \rho$  or  $(b, 0) \in \rho$  ( $a, b \in A$ ).

Then  $\rho$  is called a semi-prime congruence.

(3) If  $\rho$  satisfies the following condition:

for a congruence  $\tau : \rho \subset \tau \subset A \times A \Rightarrow \tau = \rho$  or  $\tau = A \times A$ .

Then  $\rho$  is called a maximal congruence.

(4) If  $\rho$  satisfies the following condition:

for  $a \in A$ ,  $(a, 0) \notin \rho \Rightarrow (ab, 1) \in \rho$  for some  $b \in A$ .

Then  $\rho$  is called a semi-maximal congruence.

We write

$$\text{Spec}^c(A) = \{\rho : \rho \text{ is a prime congruence of } A\},$$

$$\text{Spec}^{c'}(A) = \{\rho : \rho \text{ is a semi-prime congruence of } A\},$$

$$\text{Max}^c(A) = \{\rho : \rho \text{ is a maximal congruence of } A\},$$

$$\text{Max}^{c'}(A) = \{\rho : \rho \text{ is a semi-maximal congruence of } A\}.$$

And call  $\text{Spec}^c(A)$  (resp.  $\text{Spec}^{c'}(A)$ ,  $\text{Max}^c(A)$  or  $\text{Max}^{c'}(A)$ ) the spectrum of prime (resp. semi-prime, maximal or semi-maximal) congruences of  $A$ . Obviously,  $\text{Spec}^c(A) \subset \text{Spec}^{c'}(A)$ .

**Lemma 2.2.** Let  $A$  be a commutative semiring.

- (1) If  $\{\rho_\alpha\}_{\alpha \in \Lambda}$  is a family of congruences of  $A$ , then  $\bigcap_{\alpha \in \Lambda} \rho_\alpha$  is a congruence of  $A$ .
- (2) For a congruence  $\rho \neq A \times A$  of  $A$ ,  $\rho$  is a semi-prime congruence  $\Leftrightarrow A/\rho$  is a semidomain.
- (3) For a congruence  $\rho \neq A \times A$  of  $A$ ,  $\rho$  is a semi-maximal congruence  $\Leftrightarrow A/\rho$  is a semifield.
- (4) Any congruence  $\rho \neq A \times A$  is contained in a maximal congruence of  $A$ . In particular,  $A$  contains a maximal congruence.

**Proof.** (1), (2) and (3) follow directly from the definitions, and (4) is proved by using Zorn's lemma.  $\square$

By Lemma 2.2(2),(3) above, a semi-maximal congruence is also a semi-prime congruence. By Lemma 2.2(1), for any relation  $R$  on  $A$ , i.e.,  $R \subset A \times A$ , there exists the unique smallest congruence of  $A$  containing  $R$ , which is the intersection of all congruences of  $A$  containing  $R$ . We denote it by  $R^c$ , and call it the congruence

generated by  $R$ .

**Definition 2.3.** Let  $A$  be a commutative semiring. We define an operation  $*$  on the Cartesian product  $A \times A$  as follows:

$$(a, b) * (c, d) = (ac + bd, ad + bc) \quad (\forall (a, b), (c, d) \in A \times A).$$

Then for  $(a, b) \in A \times A$  and every positive integer  $n$ , we define  $(a, b)^{*n}$  inductively as follows:

$$(a, b)^{*1} = (a, b), \quad (a, b)^{*2} = (a, b) * (a, b), \quad \dots, \quad (a, b)^{*n} = (a, b)^{*(n-1)} * (a, b).$$

We also define  $(a, b)^{*0} = (1, 0)$ .

Using this operation, the condition for a congruence  $\rho$  of  $A$  to be a prime congruence is as follows:  $(a, b) * (c, d) \in \rho \Rightarrow (a, b) \in \rho$  or  $(c, d) \in \rho$  ( $a, b, c, d \in A$ ).

In the following, for two non-empty subsets  $S$  and  $T$  of  $A \times A$ , we shall write  $S * T = \{\alpha \in A \times A : \alpha = (a, b) * (c, d) \text{ with } (a, b) \in S, (c, d) \in T\}$ .

**Lemma 2.4.** Let  $A$  be a commutative semiring.

- (1)  $*$  is associative:  $((a, b) * (c, d)) * (e, f) = (a, b) * ((c, d) * (e, f))$ ;
- (2)  $*$  is commutative:  $(a, b) * (c, d) = (c, d) * (a, b)$ ;
- (3)  $*$  is distributive:  $((a, b) + (c, d)) * (e, f) = (a, b) * (e, f) + (c, d) * (e, f)$ ;
- (4)  $(a, b) * (c, c) \in \text{id}_A$ ;
- (5)  $(a, b) * (1, 0) = (a, b)$ ,  $(a, b) * (0, 1) = (b, a)$ ,  $(a, b) * (0, 0) = (0, 0)$ .
- (6) for positive integers  $n$ , we have
$$(b, a)^{*n} = \begin{cases} (a, b)^{*n} & \text{if } 2 \mid n \\ (a, b)^{*n} * (0, 1) & \text{if } 2 \nmid n. \end{cases}$$
In particular,  $(0, 1)^{*n} = \begin{cases} (1, 0) & \text{if } 2 \mid n \\ (0, 1) & \text{if } 2 \nmid n. \end{cases}$ 
 $(a, b, c, d, e, f \in A).$

**Proof.** (1)~(5) follow directly from the definitions, and (6) is proved by

induction.  $\square$

**Corollary 2.5.** Let  $A$  be a commutative semiring. Then  $(A \times A, +, *)$  is a commutative semiring with additive zero  $(0, 0)$  and multiplicative unity  $(1, 0)$ . Moreover, if  $\rho$  is a congruence on  $A$ , then  $\rho$  is an ideal of  $(A \times A, +, *)$ .

**Proof.** This follows from Lemma 2.4 above.  $\square$

**Proposition 2.6.** Let  $A$  be a commutative semiring, and  $n$  be a positive integer. Then for  $(a, b), (c, d) \in A \times A$ , we have

$$\begin{aligned} (a, b)^{*n} &= \left( \sum_{2|i, i=0}^n \binom{n}{i} a^{n-i} b^i, \sum_{2|i, i=1}^n \binom{n}{i} a^{n-i} b^i \right), \\ ((a, b) + (c, d))^{*n} &= (a + c, b + d)^{*n} = \sum_{i=0}^n \binom{n}{i} (a, b)^{*i} * (c, d)^{*(n-i)}. \end{aligned}$$

**Proof.** We use induction on  $n$ . The case for  $n = 1$  is obvious. For the first equality, assume it holds for  $n$ , then

$$\begin{aligned} (a, b)^{*(n+1)} &= (a, b) * (a, b)^{*n} = (a, b) * \left( \sum_{2|i, i=0}^n \binom{n}{i} a^{n-i} b^i, \sum_{2|i, i=1}^n \binom{n}{i} a^{n-i} b^i \right) \\ &= \left( \sum_{2|i, i=0}^n \binom{n}{i} a^{n+1-i} b^i + \sum_{2|i, i=1}^n \binom{n}{i} a^{n-i} b^{i+1}, \sum_{2|i, i=0}^n \binom{n}{i} a^{n-i} b^{i+1} \right. \\ &\quad \left. + \sum_{2|i, i=1}^n \binom{n}{i} a^{n+1-i} b^i \right) \\ &= \left( \sum_{2|i, i=0}^n \binom{n+1}{i} a^{n+1-i} b^i + \frac{1 + (-1)^{n+1}}{2} b^{n+1}, \sum_{2|i, i=1}^n \binom{n+1}{i} a^{n+1-i} b^i \right. \\ &\quad \left. + \frac{1 - (-1)^{n+1}}{2} b^{n+1} \right) \\ &= \left( \sum_{2|i, i=0}^{n+1} \binom{n+1}{i} a^{n+1-i} b^i, \sum_{2|i, i=1}^{n+1} \binom{n+1}{i} a^{n+1-i} b^i \right). \end{aligned}$$

So via the induction, the first equality holds for all positive integers  $n$ .

For the second equality, assume it holds for  $n$ , then

$$\begin{aligned}
((a, b) + (c, d))^{*(n+1)} &= ((a, b) + (c, d)) * \sum_{i=0}^n \binom{n}{i} (a, b)^{*i} * (c, d)^{*(n-i)} \\
&= \sum_{i=0}^n \binom{n}{i} (a, b)^{*(i+1)} * (c, d)^{*(n-i)} + \sum_{i=0}^n \binom{n}{i} (a, b)^{*i} * (c, d)^{*(n+1-i)} \\
&= \sum_{i=1}^{n+1} \binom{n}{i-1} (a, b)^{*i} * (c, d)^{*(n+1-i)} + \sum_{i=0}^n \binom{n}{i} (a, b)^{*i} * (c, d)^{*(n+1-i)} \\
&= \sum_{i=1}^n \binom{n+1}{i} (a, b)^{*i} * (c, d)^{*(n+1-i)} + \binom{n+1}{n+1} (a, b)^{*(n+1)} * (c, d)^{*0} \\
&\quad + \binom{n+1}{0} (a, b)^{*0} * (c, d)^{*(n+1)} \\
&= \sum_{i=0}^{n+1} \binom{n+1}{i} (a, b)^{*i} * (c, d)^{*(n+1-i)}.
\end{aligned}$$

So via the induction, the second equality holds for all positive integers  $n$ .  $\square$

**Definition 2.7.** Let  $A$  be a commutative semiring and  $R$  be a relation of  $A$ .

We define  $R_+ = \{(a, b) \in A \times A : (a + c, b + c) \in R \text{ for some } c \in A\}$ .

**Lemma 2.8.** Let  $A$  be a commutative semiring and  $\rho$  be a congruence on  $A$ .

Then  $\rho_+$  is also a congruence of  $A$ , and  $\rho_+ \supset \rho$ . Moreover,  $(\rho_+)_+ = \rho_+$ .

**Proof.** Follows easily from the definition.  $\square$

**Definition 2.9.** Let  $A$  be a commutative semiring, and  $\rho$  be a congruence on

$A$ . We define

$$\sqrt{\rho} = \{(a, b) \in A \times A : (a + c, b + c)^{*n} \in \rho \text{ for some } c \in A \text{ and some positive integer } n\}.$$

Obviously,  $\rho \subset \sqrt{\rho}$ .

**Proposition 2.10.** Let  $A$  be a commutative semiring, and  $\rho$  be a congruence

on  $A$ . Then  $\sqrt{\rho}$  is also a congruence on  $A$ . Moreover,  $(\sqrt{\rho})_+ = \sqrt{\rho}$ .



**Proof.** Firstly,  $(a, b) \in \sqrt{\rho} \Leftrightarrow (b, a) \in \sqrt{\rho}$ . In fact, if  $(a, b) \in \sqrt{\rho}$ , then  $(a + c, b + c)^{*n} \in \rho$  for some  $c \in A$  and some positive integer  $n$ . So  $(a + c, b + c)^{*n} * (0, 1) \in \rho$ . Thus by Lemma 2.4 above,  $(b + c, a + c)^{*n} \in \rho$ , so  $(b, a) \in \sqrt{\rho}$ . and vice versa. Next, if  $(a, b), (b, c) \in \sqrt{\rho}$ , then  $(a + e, b + e)^{*m} \in \rho$  and  $(b + f, c + f)^{*n} \in \rho$  for some  $e, f \in A$  and some positive integers  $m, n$ . Then by Lemma 2.4 and Prop.2.6 above,

$$\begin{aligned} (a + e + f, b + e + f)^{*m} &= ((a + e, b + e) + (f, f))^m \\ &= \sum_{i=0}^m \binom{m}{i} (a + e, b + e)^{*i} * (f, f)^{*(m-i)} \\ &= (a + e, b + e)^{*m} + \sum_{i=0}^{m-1} \binom{m}{i} (a + e, b + e)^{*i} * (f, f)^{*(m-i)} \in \rho. \end{aligned}$$

Similarly,  $(b + e + f, c + e + f)^{*n} \in \rho$ . Write  $t = e + f$ , then the above discussion shows that  $(a + t, b + t)^{*m}, (b + t, c + t)^{*n} \in \rho$ . Write  $s = b + 2t \in A$  and  $k = m + n$ . For  $i = 0, 1, \dots, k$ , if  $i \geq m$ , then  $(a + t, b + t)^{*i} = (a + t, b + t)^{*m} * (a + t, b + t)^{*(i-m)} \in \rho$ . If  $i < m$ , then  $k - i > k - m = n$ , so  $(b + t, c + t)^{*(k-i)} = (b + t, c + t)^{*n} * (b + t, c + t)^{*(k-i-n)} \in \rho$ . Then

$$\begin{aligned} (a + s, c + s)^{*k} &= ((a + t, b + t) + (b + t, c + t))^k \\ &= \sum_{i=0}^k \binom{k}{i} (a + t, b + t)^{*i} * (b + t, c + t)^{*(k-i)} \in \rho. \end{aligned}$$

So  $(a, c) \in \sqrt{\rho}$ . Also  $\text{id}_A \subset \rho \subset \sqrt{\rho}$ . Therefore,  $\sqrt{\rho}$  is an equivalence relation of  $A$ .

Next, let  $(a, b), (c, d) \in \sqrt{\rho}$ . Then by definition,  $(a + e, b + e)^{*m} \in \rho$  and  $(c + f, d + f)^{*n} \in \rho$  for some  $e, f \in A$  and some positive integers  $m, n$ . Write  $k = m + n$ . By Prop.2.6 above,  $((a + e, b + e) + (c + f, d + f))^k = \sum_{i=0}^k \binom{k}{i} (a + e, b + e)^{*i} * (c + f, d + f)^{*(k-i)}$ . For  $i = 0, 1, \dots, k$ , if  $i \geq m$ , then  $(a + e, b + e)^{*i} = (a + e, b + e)^{*m} * (a + e, b + e)^{*(i-m)} \in \rho$ ; if  $i < m$ , then  $k - i > k - m = n$ , so

$(c + f, d + f)^{(k-i)} = (c + f, d + f)^{*n} * (c + f, d + f)^{(k-i-n)} \in \rho$ , which implies that  
 $((a + e, b + e) + (c + f, d + f))^{*k} \in \rho$ , i.e.,  $((a + c + e + f, b + d + e + f))^{*k} \in \rho$ .  
 So  $(a + c, b + d) \in \sqrt{\rho}$ , i.e.,  $(a, b) + (c, d) \in \sqrt{\rho}$ . The remainder is to show that  
 $(a, b) \in \sqrt{\rho} \Rightarrow (ac, bc) \in \sqrt{\rho} \ (\forall c \in A)$ . Indeed,  $(a, b) \in \sqrt{\rho} \Rightarrow (a + e, b + e)^{*n} \in \rho$  for  
 some  $e \in A$  and some positive integer  $n$ . Then by Prop.2.6 above,

$$\begin{aligned}
 & ((a + e)c, (b + e)c)^{*n} \\
 &= \left( \sum_{2|i, i=0}^n \binom{n}{i} ((a + e)c)^{n-i} \cdot ((b + e)c)^i, \sum_{2|i, i=1}^n \binom{n}{i} ((a + e)c)^{n-i} \cdot ((b + e)c)^i \right) \\
 &= c^n \cdot \left( \sum_{2|i, i=0}^n \binom{n}{i} (a + e)^{n-i} \cdot (b + e)^i, \sum_{2|i, i=1}^n \binom{n}{i} (a + e)^{n-i} \cdot (b + e)^i \right) \\
 &= c^n \cdot (a + e, b + e)^{*n} \in \rho \text{ (here we write } x \cdot (y, z) = (xy, xz) \ (\forall x, y, z \in A)),
 \end{aligned}$$

so  $(ac + ec, bc + ec)^{*n} \in \rho$ , hence  $(ac, bc) \in \sqrt{\rho}$ . Therefore,  $\sqrt{\rho}$  is a congruence. To  
 show that  $(\sqrt{\rho})_+ = \sqrt{\rho}$ , firstly, by Lemma 2.8 above,  $\sqrt{\rho} \subset (\sqrt{\rho})_+$ . Conversely, if  
 $(a, b) \in (\sqrt{\rho})_+$ , then  $(a + c, b + c) \in \sqrt{\rho}$  for some  $c \in A$ . So  $(a + c + d, b + c + d)^{*n} \in \rho$  for  
 some  $d \in A$  and some positive integer  $n$ , i.e.,  $(a + e, b + e)^{*n} \in \rho$  with  $e = c + d \in A$ ,  
 so  $(a, b) \in \sqrt{\rho}$ , and so  $(\sqrt{\rho})_+ \subset \sqrt{\rho}$ . Therefore  $(\sqrt{\rho})_+ = \sqrt{\rho}$ , and the proof is  
 completed.  $\square$

**Definition 2.11.** Let  $A$  be a commutative semiring, and  $\rho$  be a congruence  
 on  $A$ .

- (1) If  $\sqrt{\rho} = \rho$ , then  $\rho$  is called to be a radical congruence on  $A$ .
- (2) If  $\sqrt{\rho} = \rho_+$ , then  $\rho$  is called to be a quasi-radical congruence on  $A$ .
- (3) Denote  $R_{\text{nil}}(A) = \{(a, b) \in A \times A : (a, b)^{*n} \in \text{id}_A \text{ for some positive integer } n\}$   
 and  $\rho_{\text{nil}}(A) = R_{\text{nil}}(A)_+$ , then  $\rho_{\text{nil}}(A) = \sqrt{\text{id}_A}$ , and  $\rho_{\text{nil}}(A)$  is called to be the nilpotent  
 congruence of  $A$ .

**Proposition 2.12.** Let  $A$  be a commutative semiring. Let  $\rho, \rho_1$  and  $\rho_2$  be congruences on  $A$ .

- (1)  $\rho_1 \subset \rho_2 \Rightarrow \sqrt{\rho_1} \subset \sqrt{\rho_2}$  and  $(\rho_1)_+ \subset (\rho_2)_+$ .    (2)  $\rho_+ \subset \sqrt{\rho}$ .
- (3)  $\sqrt{\sqrt{\rho}} = \sqrt{\rho_+} = \sqrt{\sqrt{\rho_+}}$ , so  $\sqrt{\rho_+}$  is a radical congruence on  $A$ . In particular, if  $\rho = \rho_+$ , then  $\sqrt{\rho}$  is a radical congruence on  $A$ .
- (4)  $\sqrt{\sqrt{\sqrt{\rho}}} = \sqrt{\sqrt{\rho}}$ , i.e.,  $\sqrt{\sqrt{\rho}}$  is a radical congruence on  $A$ .
- (5)  $(a, b), (b, c) \in R_{\text{nil}}(A) \Rightarrow (a, c)^{*k} + (e, e) \in \text{id}_A$  for some  $e \in A$  and some positive integer  $k$ .
- (6) If  $\rho$  is a prime congruence, then  $\sqrt{\rho} = \rho_+$ , i.e.,  $\rho$  is a quasi-radical congruence.

If the prime congruence  $\rho$  satisfies  $\rho = \rho_+$ , then  $\rho$  is a radical congruence.

**Proof.** (1) and (2) follow directly from the definitions.

(3) By (2),  $\rho_+ \subset \sqrt{\rho}$ , so by (1),  $\sqrt{\rho_+} \subset \sqrt{\sqrt{\rho}}$ . Conversely, if  $(a, b) \in \sqrt{\sqrt{\rho}}$ , then  $(a + e, b + e)^{*m} \in \sqrt{\rho}$  for some  $e \in A$  and some positive integer  $m$ . Write  $(a + e, b + e)^{*m} = (x, y)$ , then from  $(x, y) \in \sqrt{\rho}$  we have  $(x + f, y + f)^{*n} \in \sqrt{\rho}$  for some  $f \in A$  and some positive integer  $n$ , i.e.,  $((x, y) + (f, f))^{*n} \in \rho$ , which implies, by Prop.2.6 and Lemma 2.4 above, that  $(x, y)^{*n} + (t, t) \in \rho$  with  $(t, t) = \sum_{i=0}^{n-1} \binom{n}{i} (x, y)^{*i} * (f, f)^{*(n-i)}$ . So  $(a + e, b + e)^{*(mn)} + (t, t) \in \rho$ , and so  $(a + e, b + e)^{*(mn)} \in \rho_+$ , hence  $(a, b) \in \sqrt{\rho_+}$ . So  $\sqrt{\sqrt{\rho}} \subset \sqrt{\rho_+}$ . Therefore  $\sqrt{\sqrt{\rho}} = \sqrt{\rho_+}$ . The proof of the equality  $\sqrt{\sqrt{\rho_+}} = \sqrt{\rho_+}$  is similar.

(4) By (3),  $\sqrt{\sqrt{\rho}} = \sqrt{\rho_+}$ , so  $\sqrt{\sqrt{\sqrt{\rho}}} = \sqrt{\sqrt{\rho_+}} = \sqrt{\rho_+} = \sqrt{\sqrt{\rho}}$ .

(5) By Prop.2.6 and Lemma 2.4 above, the conclusion follows from the definition.

(6) The conclusion follows from the definition. The proof is completed.  $\square$

Recall that the semiring  $A$  satisfies the additive annihilation law if

$$a + c = b + c \Rightarrow a = b \quad (a, b, c \in A).$$

For example,  $(\mathbb{Z}_{\geq 0}, +, \cdot)$  satisfies the additive annihilation law, where  $\mathbb{Z}_{\geq 0} = \{n \in \mathbb{Z} : n \geq 0\}$ .

**Proposition 2.13.** Let  $A$  be a commutative semiring satisfying the additive annihilation law. Then  $\rho_{\text{nil}}(A) = R_{\text{nil}}(A)$ , and  $\rho_{\text{nil}}(A/\rho_{\text{nil}}(A)) = \text{id}$ .

**Proof.** By definition,  $R_{\text{nil}}(A) \subset \rho_{\text{nil}}(A)$ . Conversely, for  $(a, b) \in \rho_{\text{nil}}(A)$ , we have  $(a + c, b + c) \in R_{\text{nil}}(A)$  for some  $c \in A$ , i.e.,  $(a + c, b + c)^{*n} \in \text{id}_A$  for some positive integer  $n$ . Denote  $(a + c, b + c)^{*n} = (e, e)$  for some  $e \in A$ . By Prop.2.6 above,

$$\begin{aligned} (a + c, b + c)^{*n} &= ((a, b) + (c, c))^{*n} = \sum_{i=0}^n \binom{n}{i} (a, b)^{*i} * (c, c)^{*(n-i)} \\ &= (a, b)^{*n} + \sum_{i=0}^{n-1} \binom{n}{i} (a, b)^{*i} * (c, c)^{*(n-i)}. \end{aligned}$$

Obviously,  $\sum_{i=0}^{n-1} \binom{n}{i} (a, b)^{*i} * (c, c)^{*(n-i)} \in \text{id}_A$ , so  $\sum_{i=0}^{n-1} \binom{n}{i} (a, b)^{*i} * (c, c)^{*(n-i)} = (f, f)$  for some  $f \in A$ . If we write  $(a, b)^{*n} = (x, y)$ , then  $(e, e) = (x, y) + (f, f)$ , so  $x + f = e = y + f$ , and so  $x = y$  via the additive annihilation law. Then  $(a, b)^{*n} = (x, x) \in \text{id}_A$ , so  $(a, b) \in R_{\text{nil}}(A)$ , which implies that  $\rho_{\text{nil}}(A) \subset R_{\text{nil}}(A)$ , and the first equality holds.

Next to show the second equality, we first write  $\overline{A} = A/\rho_{\text{nil}}(A)$ , the quotient semiring. Then by Prop.2.10 above,  $\rho_{\text{nil}}(\overline{A})$  is a congruence of  $\overline{A}$ , in particular,  $\rho_{\text{nil}}(\overline{A}) \supset \text{id}_{\overline{A}}$ . For  $(\overline{x}, \overline{y}) \in \rho_{\text{nil}}(\overline{A})$  with  $x, y \in A$ , we have  $(\overline{x} + \overline{c}, \overline{y} + \overline{c}) \in R_{\text{nil}}(\overline{A})$  for some  $c \in A$ , so  $(\overline{x} + \overline{c}, \overline{y} + \overline{c})^{*n} \in \text{id}_{\overline{A}}$  for some positive integer  $n$ , and we write  $(\overline{x} + \overline{c}, \overline{y} + \overline{c})^{*n} = (\overline{e}, \overline{e})$

with  $e \in A$ . By Prop.2.6 above,

$$\begin{aligned} (\bar{x} + \bar{c}, \bar{y} + \bar{c})^{*n} &= \sum_{i=0}^n \binom{n}{i} (\bar{x}, \bar{y})^{*i} * (\bar{c}, \bar{c})^{*(n-i)} \\ &= (\bar{x}, \bar{y})^{*n} + \sum_{i=0}^{n-1} \binom{n}{i} (\bar{x}, \bar{y})^{*i} * (\bar{c}, \bar{c})^{*(n-i)}. \end{aligned}$$

Obviously,  $\sum_{i=0}^{n-1} \binom{n}{i} (\bar{x}, \bar{y})^{*i} * (\bar{c}, \bar{c})^{*(n-i)} \in \text{id}_{\bar{A}}$ , so we may write  $\sum_{i=0}^{n-1} \binom{n}{i} (\bar{x}, \bar{y})^{*i} * (\bar{c}, \bar{c})^{*(n-i)} = (\bar{f}, \bar{f})$  with  $f \in A$ . Then if we write  $(\bar{x}, \bar{y})^{*n} = (\bar{u}, \bar{v})$  with  $u, v \in A$ , we have  $(\bar{e}, \bar{e}) = (\bar{u}, \bar{v}) + (\bar{f}, \bar{f})$ , so  $\overline{u+f} = \bar{e} = \overline{v+f}$ , and so  $(u+f, v+f) \in \rho_{\text{nil}}(A) = R_{\text{nil}}(A)$  via the first equality. Hence by definition,  $(u, v) \in \rho_{\text{nil}}(A)$ , so  $\bar{u} = \bar{v} \in \bar{A}$ , i.e.,  $(\bar{x}, \bar{y})^{*n} = (\bar{u}, \bar{v}) \in \text{id}_{\bar{A}}$ . By Prop.2.6 above,

$$(\bar{x}, \bar{y})^{*n} = \left( \sum_{2|i, i=0}^n \binom{n}{i} \bar{x}^{n-i} \bar{y}^i, \sum_{2 \nmid i, i=1}^n \binom{n}{i} \bar{x}^{n-i} \bar{y}^i \right).$$

$$\text{So } \overline{\sum_{2|i, i=0}^n \binom{n}{i} x^{n-i} y^i} = \overline{\sum_{2 \nmid i, i=1}^n \binom{n}{i} x^{n-i} y^i} \text{ in } \bar{A}. \text{ Hence}$$

$$(x, y)^{*n} = \left( \sum_{2|i, i=0}^n \binom{n}{i} x^{n-i} y^i, \sum_{2 \nmid i, i=1}^n \binom{n}{i} x^{n-i} y^i \right) \in \rho_{\text{nil}}(A) = R_{\text{nil}}(A)$$

via the first equality, which implies that  $((x, y)^{*n})^{*m} \in \text{id}_A$  for some positive integer  $m$ , i.e.,  $(x, y)^{*(mn)} \in \text{id}_A$ , so  $(x, y) \in R_{\text{nil}}(A) = \rho_{\text{nil}}(A)$ , and then  $\bar{x} = \bar{y}$  in  $\bar{A}$ , i.e.,  $(\bar{x}, \bar{y}) \in \text{id}_{\bar{A}}$ . So  $\rho_{\text{nil}}(\bar{A}) \subset \text{id}_{\bar{A}}$ , which shows the second equality, and the proof is completed.  $\square$

For a commutative semiring  $A$ , if  $R_{\text{nil}}(A) = \text{id}_A$ , then  $A$  is called to be reduced. If  $\rho_{\text{nil}}(A) = \text{id}_A$ , then  $A$  is called to be strongly reduced. obviously, strongly reduced  $\Rightarrow$  reduced. We denote  $\mathfrak{N}_c(A) = R_{\text{nil}}(A)^c$ , which is the congruence of  $A$  generated by the relation  $R_{\text{nil}}(A)$ , and call  $\mathfrak{N}_c(A)$  the quasi-nilpotent congruence of  $A$ . Obviously,  $\rho_{\text{nil}}(A) \supset \mathfrak{N}_c(A)$ , also,  $A$  is reduced if and only if  $\mathfrak{N}_c(A) = \text{id}_A$ .

Now for two relations  $R$  and  $R'$  on a non-empty set  $S$ , recall that their product

$$R \circ R' = \{(a, c) \in S \times S : \exists b \in S \text{ such that } (a, b) \in R \text{ and } (b, c) \in R'\},$$

and the inverse  $R^{-1} = \{(a, b) \in S \times S : (b, a) \in R\}$  (see [Ho, pp.14, 15]). Obviously, for relations  $R, R_1, \dots, R_n, R', R''$  on  $S$ , one has  $(R^{-1})^{-1} = R$ ;  $(R_1 \circ \dots \circ R_n)^{-1} = R_n^{-1} \circ \dots \circ R_1^{-1}$ ;  $R \subset R' \Rightarrow R^{-1} \subset R'^{-1}$ ,  $R \circ R'' \subset R' \circ R''$  and  $R'' \circ R \subset R'' \circ R'$ .

As in [Ho], for any relation  $R$  on  $S$ , we denote its transitive closure by  $R^\infty$ , which is defined by  $R^\infty = \bigcup_{n=1}^\infty R^n$ , where  $R^n = \underbrace{R \circ \dots \circ R}_n$ . It is well known that  $R^\infty$  is the smallest transitive relation on  $S$  containing  $R$  (see [Ho, p.20]). Also, we denote by  $R^e$  the equivalence on  $S$  generated by  $R$ , i.e., the smallest equivalence on  $S$  containing  $R$ . Then  $R^e = (R \cup R^{-1} \cup \text{id}_S)^\infty$  (see [Ho, p.20]). Moreover,  $(x, y) \in R^e \Leftrightarrow x = y$  or for some positive integer  $n$  there exists a sequence  $x = z_1 \rightarrow z_2 \rightarrow \dots \rightarrow z_n = y$  in which for each  $i \in \{1, 2, \dots, n-1\}$  either  $(z_i, z_{i+1}) \in R$  or  $(z_{i+1}, z_i) \in R$  (see [Ho, p.21]).

**Definition 2.14.** Let  $A$  be a commutative semiring,  $R$  be a relation on  $A$ , and  $E$  be an equivalence relation on  $A$ . We define the relations

$$R^L = \{(ax + y, bx + y) : (a, b) \in R, x, y \in A\}, \text{ and}$$

$$E^b = \{(a, b) \in A \times A : (ax + y, bx + y) \in E \text{ for all } x, y \in A\}.$$

Since  $0, 1 \in A$ , we have  $R \subset R^L$ . Also,  $\rho^L = \rho$  for any congruence  $\rho$  of  $A$ .

**Lemma 2.15.** Let  $A$  be a commutative semiring,  $R_1$  and  $R_2$  be two relations on  $A$ , and  $E$  be an equivalence relation on  $A$ . We have

- (1)  $R_1 \subset R_1^L$ ; (2)  $(R_1^L)^{-1} = (R_1^{-1})^L$ ; (3)  $R_1 \subset R_2 \Rightarrow R_1^L \subset R_2^L$ ;
- (4)  $(R_1^L)^L = R_1^L$ ; (5)  $(R_1 \cup R_2)^L = R_1^L \cup R_2^L$ ;

(6)  $R_1 = R_1^L$  if and only if  $R_1$  satisfies the following condition:

$$(a, b) \in R_1 \Rightarrow (ax, bx), (a + y, b + y) \in R_1 \quad (\forall x, y \in A).$$

(7)  $R_1 = R_1^L \Rightarrow R_1^n = (R_1^L)^n$  for all positive integer  $n$ .

(8)  $E^\flat$  is the largest congruence on  $A$  contained in  $E$ .

**Proof.** Follow easily from the definitions.  $\square$

**Proposition 2.16.** Let  $R$  be a relation on a commutative semiring  $A$ . Then  $R^c = (R^L)^e$ , where  $R^c$  is the congruence of  $A$  generated by  $R$  as before.

**Proof.** By definition,  $(R^L)^e$  is the equivalence on  $A$  generated by  $R^L$ . Denote  $S = R^L \cup (R^L)^{-1} \cup \text{id}_A$ , then  $(R^L)^e = S^\infty$  (see [Ho, p.20]). By Lemma 2.15 above,  $S = (R \cup R^{-1} \cup \text{id}_A)^L$ , so  $S^L = S$ , and so  $S^n = (S^L)^n$  for all positive integers  $n$ . Now for any  $(a, b) \in (R^L)^e$ , by the above discussion,  $(a, b) \in S^n$  for some positive integer  $n$ . Since  $S^n = (S^L)^n$ , by Lemma 2.15 above,  $(ax, bx) \in S^n$  and  $(a + y, b + y) \in S^n$ , hence  $(ax, bx), (a + y, b + y) \in (R^L)^e$  ( $\forall x, y \in A$ ). In particular, if  $(a, b), (c, d) \in (R^L)^e$ , then  $(a + c, b + c) \in (R^L)^e$  and  $(b + c, b + d) \in (R^L)^e$ , so  $(a + c, b + d) \in (R^L)^e$ . Hence  $(R^L)^e$  is a congruence of  $A$  containing  $R$ . Now let  $\rho$  be a congruence of  $A$  containing  $R$ . Then by Lemma 2.15 above,  $R^L \subset \rho^L = \rho$ . So  $(R^L)^e \subset \rho^e = \rho$ , which shows that  $R^c = (R^L)^e$ , and the proof is completed.  $\square$

**Corollary 2.17.** Let  $R$  be a relation on a commutative semiring  $A$ . Then for  $a, b \in A$ ,  $(a, b) \in R^c$  if and only if  $a = b$  or for some positive integer  $n$  there exists a sequence  $a = z_1 \rightarrow z_2 \rightarrow \cdots \rightarrow z_n = b$  in which for each  $i \in \{1, 2, \dots, n - 1\}$  either  $(z_i, z_{i+1}) \in R^L$  or  $(z_{i+1}, z_i) \in R^L$ .

**Proof.** Follows from the above Prop.2.16 (see [Ho, p.21]).  $\square$

Let  $R$  and  $\rho$  be a relation and a congruence on a commutative semiring  $A$ , respectively. Define the relation

$$R/\rho = \{(a\rho, b\rho) : (a, b) \in R\} = \{(\bar{a}, \bar{b}) \in A/\rho \times A/\rho : (a, b) \in R\}$$

on the quotient semiring  $A/\rho$ . If  $\text{id}_A \subset R \subset \rho$ , then obviously  $R/\rho = \text{id}_{\bar{A}}$  with  $\bar{A} = A/\rho$ .

**Proposition 2.18.** Let  $R$  and  $\rho$  be a relation and a congruence on a commutative semiring  $A$ , respectively. Then  $R^c/\rho = (R/\rho)^c$  on the quotient semiring  $\bar{A} = A/\rho$ .

**Proof.** Obviously,  $R^c/\rho$  is a congruence on  $\bar{A}$  containing  $R/\rho$ . So  $R^c/\rho \supset (R/\rho)^c$ . Conversely, let  $(\bar{a}, \bar{b}) \in R^c/\rho$  with  $(a, b) \in R^c$ . Then by Cor.2.17 above, either  $a = b$  or for some positive integer  $n$  there exists a sequence  $a = z_1 \rightarrow z_2 \rightarrow \cdots \rightarrow z_n = b$  in which for each  $i \in \{1, 2, \dots, n-1\}$  either  $(z_i, z_{i+1}) \in R^L$  or  $(z_{i+1}, z_i) \in R^L$ . i.e.,  $(z_i, z_{i+1})$  or  $(z_{i+1}, z_i) = (a_i x_i + y_i, b_i x_i + y_i)$  for some  $x_i, y_i \in A$  and  $(a_i, b_i) \in R$ . Hence in  $\bar{A}$ ,  $(\bar{z}_i, \bar{z}_{i+1})$  or  $(\bar{z}_{i+1}, \bar{z}_i) = (\bar{a}_i \bar{x}_i + \bar{y}_i, \bar{b}_i \bar{x}_i + \bar{y}_i)$  with  $(\bar{a}_i, \bar{b}_i) \in R/\rho$  and  $\bar{x}_i, \bar{y}_i \in \bar{A}$ . That is,  $(\bar{z}_i, \bar{z}_{i+1})$  or  $(\bar{z}_{i+1}, \bar{z}_i) \in (R/\rho)^L$ . So by Cor.2.17 above, we get  $(\bar{a}, \bar{b}) \in (R/\rho)^c$ . Therefore  $R^c/\rho = (R/\rho)^c$ , and the proof is completed.  $\square$

Recall that for a non-empty subset  $T$  of a partially ordered set  $(S, \leq)$ , an element  $s \in S$  is a lower (resp. upper) bound for  $T$  if  $s \leq t$  (resp.  $t \leq s$ ) for every  $t \in T$ . If the set of lower (resp. upper) bounds of  $T$  is non-empty and has a maximum (resp. least) element  $d$ , we refer to  $d$  as the greatest lower bound (resp. least upper bound) or meet (resp. join) of  $T$ . The element  $d$  is unique if it exists. We write  $d = \wedge\{t : t \in T\}$  (resp.  $d = \vee\{t : t \in T\}$ ) (see [Ho, p.12]). If  $T = \{a, b\}$ , then we



write  $d = a \wedge b$  (resp.  $d = a \vee b$ ). If  $(S, \leq)$  is such that  $a \wedge b$  (resp.  $a \vee b$ ) exists for every pair  $a, b \in S$ , then  $(S, \leq)$  is said to be a lower (resp. upper) semilattice. If we have the stronger property that  $\bigwedge\{t : t \in T\}$  (resp.  $\bigvee\{t : t \in T\}$ ) exists for every non-empty subset  $T$  of  $S$ , then we say that  $(S, \leq)$  is a complete lower (resp. upper) semilattice. If  $(S, \leq)$  is both a (complete) lower semilattice and a (complete) upper semilattice, we call it a (complete) lattice. A lattice  $L$  with partial order  $\leq$  in which the greatest lower bound of  $a$  and  $b$  is  $a \wedge b$  and the smallest upper bound of  $a$  and  $b$  is  $a \vee b$  is written  $L = (L, \leq, \wedge, \vee)$ . By a sublattice of  $L$  we shall mean a non-empty subset  $M$  of  $L$  with the property that  $a, b \in M \Rightarrow a \wedge b, a \vee b \in M$  (see [Ho, p.12]).

Now for a commutative semiring  $A$ , we let  $\mathcal{E}(A)$  and  $\mathcal{C}(A)$  denote the set of all equivalences and the set of all congruences on  $A$ , respectively. Then both  $\mathcal{E}(A)$  and  $\mathcal{C}(A)$  are partially ordered by inclusion, i.e.,  $(\mathcal{E}(A), \subset)$  and  $(\mathcal{C}(A), \subset)$  are partially ordered sets. In fact, they are lattices: if  $\rho, \sigma \in \mathcal{E}(A)$ , then  $\rho \cap \sigma \in \mathcal{E}(A)$  is their greatest lower bound, i.e.,  $\rho \wedge \sigma = \rho \cap \sigma$ , while  $(\rho \cup \sigma)^e$  is their least upper bound, i.e.,  $\rho \vee \sigma = (\rho \cup \sigma)^e$ . Similarly, for  $\rho, \sigma \in \mathcal{C}(A)$ ,  $\rho \wedge \sigma = \rho \cap \sigma$  and  $\rho \vee \sigma = (\rho \cup \sigma)^c$ . Moreover, for  $\rho, \sigma \in \mathcal{C}(A)$ , by Lemma 2.15 above,  $(\rho \cup \sigma)^L = \rho^L \cup \sigma^L = \rho \cup \sigma$ , so by Prop.2.16 above,  $(\rho \cup \sigma)^c = ((\rho \cup \sigma)^L)^e = (\rho \cup \sigma)^e$ , i.e., the join of  $\rho$  and  $\sigma$  in  $\mathcal{C}(A)$  coincides with their join in  $\mathcal{E}(A)$ . Furthermore, for any non-empty subset  $\Omega$  of  $\mathcal{E}(A)$  (resp.  $\mathcal{C}(A)$ ), we have  $\bigwedge\{\rho : \rho \in \Omega\} = \bigcap\{\rho : \rho \in \Omega\} \in \mathcal{E}(A)$  and  $\bigvee\{\rho : \rho \in \Omega\} = (\bigcup\{\rho : \rho \in \Omega\})^e \in \mathcal{E}(A)$  (resp.  $\bigwedge\{\rho : \rho \in \Omega\} = \bigcap\{\rho : \rho \in \Omega\} \in \mathcal{C}(A)$  and  $\bigvee\{\rho : \rho \in \Omega\} = (\bigcup\{\rho : \rho \in \Omega\})^c \in \mathcal{C}(A)$ ). Hence both  $\mathcal{E}(A)$  and  $\mathcal{C}(A)$  are complete lattices. Both of them have the maximum element  $A \times A$  and minimum element  $\text{id}_A$ .

**Proposition 2.19.** Let  $\rho$  and  $\sigma$  be two congruences on a commutative semiring  $A$ . Then  $\rho \vee \sigma = (\rho \circ \sigma)^\infty$ . In other words, for  $a, b \in A$ ,  $(a, b) \in \rho \vee \sigma$  if and only if for some positive integer  $n$  there exist elements  $x_1, x_2, \dots, x_{2n-1} \in A$  such that  $(a, x_1) \in \rho, (x_1, x_2) \in \sigma, (x_2, x_3) \in \rho, \dots, (x_{2n-2}, x_{2n-1}) \in \rho, (x_{2n-1}, b) \in \sigma$ .

**Proof.** By Prop.2.16 above,  $\rho \vee \sigma = (\rho \cup \sigma)^c = (\rho \cup \sigma)^e$ . Write  $R = (\rho \cup \sigma) \cup (\rho \cup \sigma)^{-1} \cup \text{id}_A$ , then obviously  $R = \rho \cup \sigma$ , and so  $(\rho \cup \sigma)^e = R^\infty$ . Also  $\rho \circ \sigma \subset (\rho \cup \sigma)^2$ , hence  $(\rho \circ \sigma)^n \subset (\rho \cup \sigma)^{2n}$  for every positive integer  $n$ , and so  $(\rho \circ \sigma)^\infty \subset (\rho \cup \sigma)^\infty = R^\infty = (\rho \cup \sigma)^e$ . Conversely,  $\rho, \sigma \subset \rho \circ \sigma$  since both  $\rho$  and  $\sigma$  are also equivalences, and so  $\rho \cup \sigma \subset \rho \circ \sigma$ . It follows that  $(\rho \cup \sigma)^\infty \subset (\rho \circ \sigma)^\infty$ . So  $\rho \vee \sigma = R^\infty = (\rho \cup \sigma)^\infty = (\rho \circ \sigma)^\infty$ , and the proof is completed.  $\square$

**Corollary 2.20.** If  $\rho$  and  $\sigma$  are two congruences on a commutative semiring  $A$  satisfying  $\rho \circ \sigma = \sigma \circ \rho$ . Then  $\rho \vee \sigma = \rho \circ \sigma$ .

**Proof.** If  $\rho \circ \sigma = \sigma \circ \rho$ , then  $(\rho \circ \sigma)^2 = \rho^2 \circ \sigma^2 = \rho \circ \sigma$ , and so  $(\rho \circ \sigma)^n = \rho \circ \sigma$  for every positive integer  $n$ , and the conclusion then follows from Prop.2.19 above.  $\square$

**Example.** If  $A$  is a semifield, then it is easy to see that  $\rho \circ \sigma = \sigma \circ \rho$  for any two congruences  $\rho$  and  $\sigma$  on  $A$ . So  $\rho \vee \sigma = \rho \circ \sigma$ .

**Proposition 2.21.** Let  $A$  be a commutative semiring. For  $a, b \in A$ , denote  $R(a, b) = \{(ax + by + z, bx + ay + z) : x, y, z \in A\}$ . Then  $R(a, b)_+$  (as defined in Def.2.7 above) is a congruence on  $A$ , and  $\rho \subset R(a, b)_+ \subset \rho_+$  with  $\rho = \{(a, b)\}^c$ . Moreover,  $R(a, b) = (a, b) * (A \times A) + \text{id}_A$ .

**Proof.** Obviously,  $\text{id}_A \subset R(a, b)_+$ . Also, if  $(u, v) \in R(a, b)_+$ , then  $(u+s, v+s) \in$

$R(a, b)$  for some  $s \in A$ , so  $(u+s, v+s) = (ax+by+z, bx+ay+z)$  for some  $x, y, z \in A$ ,  
 so  $(v+s, u+s) = (ay+bx+z, by+ax+z) \in R(a, b)$ , so  $(v, u) \in R(a, b)_+$ . Now let  
 $(u, v), (v, w) \in R(a, b)_+$ . Then  $(u+s, v+s), (v+t, w+t) \in R(a, b)$  for some  $s, t \in A$ ,  
 so  $(u+s, v+s) = (ax+by+z, bx+ay+z)$  and  $(v+t, w+t) = (ax'+by'+z', bx'+ay'+z')$   
 for some  $x, y, z, x', y', z' \in A$ . Then  $(u+s+t, v+s+t) = (ax+by+z+t, bx+ay+z+t)$ ,  
 and  $(v+s+t, w+s+t) = (ax'+by'+z'+s, bx'+ay'+z'+s)$ . Write  $d = v+s+t =$   
 $bx+ay+z+t = ax'+by'+z'+s$ , then  $u+s+t+d = a(x+x')+b(y+y')+z+z'+s+t$ , and  
 $w+s+t+d = b(x+x')+a(y+y')+z+z'+s+t$ . So  $(u+s+t+d, w+s+t+d) \in R(a, b)$ ,  
 and so  $(u, w) \in R(a, b)_+$ . Therefore,  $R(a, b)_+$  is an equivalence on  $A$ . Next, let  
 $(u, v), (u', v') \in R(a, b)_+$ , then  $(u+s, v+s), (u'+s', v'+s') \in R(a, b)$  for some  
 $s, s' \in A$ . So  $(u+s, v+s) = (ax+by+z, bx+ay+z)$  and  $(u'+s', v'+s') =$   
 $(ax'+by'+z', bx'+ay'+z')$  for some  $x, y, z, x', y', z' \in A$ . Hence  $(u+u'+s+s', v+$   
 $v'+s+s') = (a(x+x')+b(y+y')+z+z', b(x+x')+a(y+y')+z+z') \in R(a, b)$ ,  
 so  $(u, v) + (u', v') = (u+u', v+v') \in R(a, b)_+$ . Moreover, for  $(u, v), s, x, y, z$  above,  
 let  $t \in A$ , then  $(ut+st, vt+st) = (a(tx)+b(ty)+tz, b(tx)+a(ty)+tz) \in R(a, b)$ ,  
 so  $(ut, vt) \in R(a, b)_+$ . Therefore,  $R(a, b)_+$  is a congruence on  $A$ . The remainder can  
 be verified directly, and the proof is completed.  $\square$

**Lemma 2.22.** Let  $A$  be a commutative semiring and  $\rho$  be a congruence on  
 $A$ . let  $(a, b), (c, d) \in A \times A$  and  $c = z_1 \rightarrow z_2 \rightarrow \cdots \rightarrow z_n = d$  ( $n \geq 2$ ) be a sequence  
 in  $A$ . If  $(a, b) * (z_i, z_{i+1}) \in \rho_+$  for all  $i \in \{1, 2, \dots, n-1\}$ , then  $(a, b) * (c, d) \in \rho_+$ .

**Proof.** By Lemma 2.8 above,  $\rho_+$  is a congruence on  $A$ , and  $(\rho_+)_+ = \rho_+$ .  
 Write  $t = az_2 + bz_2$ , then from  $(a, b) * (z_1, z_2) \in \rho_+$  and  $(a, b) * (z_2, z_3) \in \rho_+$ , we have  
 $(az_1 + bz_3 + t, az_3 + bz_1 + t) = (a, b) * (z_1, z_2) + (a, b) * (z_2, z_3) \in \rho_+$ , so  $(az_1 + bz_3, az_3 +$

$bz_1) \in (\rho_+)_+ = \rho_+$ , i.e.,  $(a, b) * (z_1, z_3) \in \rho_+$ . Then, from  $(a, b) * (z_1, z_3) \in \rho_+$  and  $(a, b) * (z_3, z_4) \in \rho_+$ , similarly, we have  $(a, b) * (z_1, z_4) \in \rho_+$ , and so on. After finite steps, we get  $(a, b) * (z_1, z_n) \in \rho_+$ , i.e.,  $(a, b) * (c, d) \in \rho_+$ . The proof is completed.  $\square$

**Proposition 2.23.** Let  $A$  be a commutative semiring and  $\rho$  be a maximal congruence on  $A$ . If  $\rho = \rho_+$ , then  $\rho$  is also a prime congruence.

**Proof.** Let  $(a, b), (c, d) \in A \times A$  be such that  $(a, b) * (c, d) \in \rho$ . If  $(c, d) \notin \rho$ , then we need to show that  $(a, b) \in \rho$ . To see this, let  $R = \{(c, d)\} \cup \rho$ , then  $R$  is a relation on  $A$  and  $R \supsetneq \rho$ . So  $R^c \supsetneq \rho$ , and so  $R^c = A \times A$  because  $\rho$  is maximal. In particular,  $(1, 0) \in R^c$ . By Cor.2.17 above, there exists a sequence  $1 = z_1 \rightarrow z_2 \rightarrow \cdots \rightarrow z_n = 0$  in which for each  $i \in \{1, 2, \dots, n-1\}$  either  $(z_i, z_{i+1}) \in R^L$  or  $(z_{i+1}, z_i) \in R^L$ . By Lemma 2.15 above,  $R^L = (\{(c, d)\} \cup \rho)^L = \{(c, d)\}^L \cup \rho^L = \{(c, d)\}^L \cup \rho$  because  $\rho^L = \rho$ . By definition,  $\{(c, d)\}^L = \{(cx+y, dx+y) : x, y \in A\}$ . So  $R^L = \{(cx+y, dx+y) : x, y \in A\} \cup \rho$ . As above, for each  $i \in \{1, 2, \dots, n-1\}$  either  $(z_i, z_{i+1}) \in R^L$  or  $(z_{i+1}, z_i) \in R^L$ . If  $(z_i, z_{i+1}) \in R^L$ , then  $(z_i, z_{i+1}) \in \rho$  or  $(z_i, z_{i+1}) = (cx_i + y_i, dx_i + y_i)$  for some  $x_i, y_i \in A$ . If  $(z_i, z_{i+1}) \in \rho$ , then  $(a, b) * (z_i, z_{i+1}) \in \rho$ . If  $(z_i, z_{i+1}) = (cx_i + y_i, dx_i + y_i)$ , write  $t = ay_i + by_i \in A$ , then

$$\begin{aligned} (a, b) * (z_i, z_{i+1}) &= (a, b) * (cx_i + y_i, dx_i + y_i) \\ &= (acx_i + bdx_i + t, adx_i + bcx_i + t) \\ &= ((a, b) * (c, d)) \cdot (x_i, x_i) + (t, t) \in \rho. \end{aligned}$$

So  $(z_i, z_{i+1}) \in R^L \Rightarrow (a, b) * (z_i, z_{i+1}) \in \rho$ .

Similarly,  $(z_{i+1}, z_i) \in R^L \Rightarrow (a, b) * (z_{i+1}, z_i) \in \rho$ .

Note that  $(a, b) * (z_{i+1}, z_i) = ((a, b) * (z_i, z_{i+1})) * (0, 1)$ . So the above discussion shows that  $(a, b) * (z_i, z_{i+1}) \in \rho$  for all  $i \in \{1, 2, \dots, n-1\}$ . By our assumption,  $\rho = \rho_+$ . Hence by Lemma 2.22 above, we get  $(a, b) * (1, 0) = (a, b) * (z_1, z_n) \in \rho$ , i.e.,  $(a, b) \in \rho$ . Therefore,  $\rho$  is a prime congruence on  $A$ , and the proof is completed.  $\square$

**A question.** On a commutative semiring, is a maximal congruence always also a prime congruence ?

**Definition 2.24.** Let  $A$  be a commutative semiring and  $\sigma$  be a congruence on  $A$ . We define  $V^{\text{co}}(\sigma) = \{\rho \in \text{Spec}^c(A) : \rho \supset \sigma\}$ .

**Proposition 2.25.** Let  $A$  be a commutative semiring.

- (1) If  $\sigma_1$  and  $\sigma_2$  are two congruences on  $A$ , then  $V^{\text{co}}(\sigma_1) \cup V^{\text{co}}(\sigma_2) = V^{\text{co}}(\sigma_1 \cap \sigma_2)$ .
- (2) If  $\{\sigma_\alpha\}_{\alpha \in \Lambda}$  is a family of congruences on  $A$ , then  $\bigcap_{\alpha \in \Lambda} V^{\text{co}}(\sigma_\alpha) = V^{\text{co}}(\sigma)$ , where  $\sigma = (\bigcup_{\alpha \in \Lambda} \sigma_\alpha)^c$  is the congruence on  $A$  generated by the set  $\bigcup_{\alpha \in \Lambda} \sigma_\alpha$ .
- (3)  $V^{\text{co}}(\text{id}_A) = \text{Spec}^c(A)$  and  $V^{\text{co}}(A \times A) = \emptyset$ .

**Proof.** (1) Since  $\sigma_1 \cap \sigma_2 \subset \sigma_i$  ( $i = 1, 2$ ),  $V^{\text{co}}(\sigma_1 \cap \sigma_2) \supset V^{\text{co}}(\sigma_i)$  ( $i = 1, 2$ ), so  $V^{\text{co}}(\sigma_1 \cap \sigma_2) \supset V^{\text{co}}(\sigma_1) \cup V^{\text{co}}(\sigma_2)$ . Conversely, if  $\rho \in V^{\text{co}}(\sigma_1 \cap \sigma_2)$ , then  $\rho \supset \sigma_1 \cap \sigma_2$ . If  $\rho \not\supset \sigma_1$ , equivalently,  $\rho \notin V^{\text{co}}(\sigma_1)$ , then there exists an element  $(a, b) \in \sigma_1$ , but  $(a, b) \notin \rho$ . Now for any  $(c, d) \in \sigma_2$ , we have  $(ac, ad) \in \sigma_2$ . Also  $(d, c) \in \sigma_2$ , so  $(bd, bc) \in \sigma_2$ , and then  $(ac + bd, ad + bc) \in \sigma_2$ , i.e.,  $(a, b) * (c, d) \in \sigma_2$ , also  $(a, b) * (c, d) \in \sigma_1$ , so  $(a, b) * (c, d) \in \sigma_1 \cap \sigma_2 \subset \rho$ , which implies that  $(c, d) \in \rho$  because  $\rho$  is prime. So  $\sigma_2 \subset \rho$ , i.e.,  $\rho \in V^{\text{co}}(\sigma_2)$ . This shows that  $V^{\text{co}}(\sigma_1 \cap \sigma_2) \subset V^{\text{co}}(\sigma_1) \cup V^{\text{co}}(\sigma_2)$ , and the equality holds.

- (2)  $\rho \in \bigcap_{\alpha \in \Lambda} V^{\text{co}}(\sigma_\alpha) \Leftrightarrow \rho \in V^{\text{co}}(\sigma_\alpha) \ (\forall \alpha \in \Lambda) \Leftrightarrow \rho \supset \sigma_\alpha \ (\forall \alpha \in \Lambda) \Leftrightarrow \rho \supset$

$\cup_{\alpha \in \Lambda} \sigma_\alpha \Leftrightarrow \rho \in V^{\text{co}}(\sigma)$  with  $\sigma = (\cup_{\alpha \in \Lambda} \sigma_\alpha)^c$ .

(3) Obvious. The proof is completed.  $\square$

From Prop.2.25 above, the set of  $V^{\text{co}}(\sigma)$  for all congruences  $\sigma$  on  $A$  satisfies the axiom of closed subsets, and give a topology on  $\text{Spec}^c(A)$ , which is called the Zariski topology on  $\text{Spec}^c(A)$ . A subset of  $\text{Spec}^c(A)$  is equipped with the subspace topology induced from the Zariski topology on  $\text{Spec}^c(A)$ .

### 3. congruences and ideals

Let  $A$  be a commutative semiring, recall that an ideal  $I$  of  $A$  is a non-empty subset  $I$  of  $A$  which is closed under addition and satisfies the condition that if  $a \in A$  and  $b \in I$  then  $ab \in I$ . We denote  $\mathcal{I}(A) = \{\text{all ideals of } A\}$ . Recall that  $\mathcal{C}(A) = \{\text{all congruences of } A\}$ .

For any non-empty subsets  $S$  and  $T$  of  $A$ , we denote  $S + T = \{s + t : s \in S, t \in T\}$ ,  $S \cdot T = \{st : s \in S, t \in T\}$ , and for  $a \in A$ ,  $a + S = \{a + s : s \in S\}$ ,  $a \cdot S = \{as : s \in S\}$ .

**Definition 3.1.** Let  $A$  be a commutative semiring. For  $J \in \mathcal{I}(A)$  and  $\sigma \in \mathcal{C}(A)$ , we define

$$\rho_J = \{(a, b) \in A \times A : a + J = b + J\} \text{ and } I_\sigma = \{a \in A : (a, 0) \in \sigma\}.$$

**Proposition 3.2.** Let  $A$  be a commutative semiring. We have

- (1)  $\rho_J \in \mathcal{C}(A)$  ( $\forall J \in \mathcal{I}(A)$ );    (2)  $I_\sigma \in \mathcal{I}(A)$  ( $\forall \sigma \in \mathcal{C}(A)$ );
- (3)  $I_{\rho_J} \subset J$  ( $\forall J \in \mathcal{I}(A)$ );    (4)  $\rho_{I_\sigma} \subset \sigma$  ( $\forall \sigma \in \mathcal{C}(A)$ );
- (5)  $J \subset J' \Rightarrow \rho_J \subset \rho_{J'}$  ( $J, J' \in \mathcal{I}(A)$ );    (6)  $\sigma \subset \tau \Rightarrow I_\sigma \subset I_\tau$  ( $\sigma, \tau \in \mathcal{C}(A)$ ).

**Proof.** Follow directly from the definitions.  $\square$

Note that if  $A$  is a commutative ring (obviously it is also a commutative semiring), then it is easy to see that  $I_{\rho_J} = J$  and  $\rho_{I_\sigma} = \sigma$  for all  $J \in \mathcal{I}(A)$  and  $\sigma \in \mathcal{C}(A)$ . But this property changes for a commutative semiring, i.e., the inclusions in the above Prop.3.2 (3) and (4) may be strict in general for a commutative semiring  $A$ . This can be seen from the following example.

**Example 3.3.** Let  $A = \mathbb{Z}_{\geq 0}$ , the set of non-negative integers. Then  $A$  is a commutative semiring under the usual addition and multiplication operations of integers. Let  $J = 2 \cdot \mathbb{Z}_{\geq 0} = \{2n : n \in \mathbb{Z}_{\geq 0}\}$ . Then  $J$  is an ideal of  $A$ . By definition,  $(a, b) \in \rho_J \Leftrightarrow a + J = b + J \Leftrightarrow a = b$  because  $a = \min\{a + c : c \in J\} = \min\{b + c : c \in J\} = b$ . So  $\rho_J = \text{id}_A$ . Hence  $I_{\rho_J} = \{a \in A : (a, 0) \in \rho_J\} = \{0\} \subsetneq J$ , i.e.,  $I_{\rho_J} \subsetneq J$ . Now let  $\sigma = \{(a, b) \in A \times A : a \equiv b \pmod{2}\}$ , then obviously  $\sigma$  is a congruence on  $A$ , and  $\sigma \neq \text{id}_A$ . By definition,  $I_\sigma = \{a \in A : (a, 0) \in \sigma\} = \{a \in A : a \equiv 0 \pmod{2}\} = 2 \cdot \mathbb{Z}_{\geq 0} = J$ . So by the above discussion,  $\rho_{I_\sigma} = \rho_J = \text{id}_A \subsetneq \sigma$ , i.e.,  $\rho_{I_\sigma} \subsetneq \sigma$ .  $\square$

#### 4. Zeros of polynomial congruence equations

Let  $A$  and  $B$  be two commutative semirings with  $A \subset B$ . Let  $A[x_1, \dots, x_n]$  be the commutative semiring of polynomials in  $n$  variables over  $A$  (see [G, p.3]). Let  $B^n = \{(b_1, \dots, b_n) : b_1, \dots, b_n \in B\}$  be the affine  $n$ -space over  $B$ . An element  $P \in B^n$  will be called a point, and if  $P = (b_1, \dots, b_n)$  with  $b_i \in B$ , then the  $b_i$  will be called the coordinates of  $P$ .

**Definition 4.1.** For the commutative semirings  $A \subset B$  as above, write  $S = A[x_1, \dots, x_n]$ . let  $T \subset S \times S$  be a non-empty subset, and let  $\rho$  be a congruence on

$B$ . Then we define

$$Z_\rho(T)(B) = \{P \in B^n : (f(P), g(P)) \in \rho \text{ for all } (f, g) \in T\},$$

and call  $Z_\rho(T)(B)$  the  $\rho$ -zero set of  $T$  in  $B^n$ . In particular, if  $\rho = \text{id}_B$  is the identity congruence on  $B$ , then  $Z_\rho(T)(B)$  is the set of solutions (i.e. common zeros) of the system of polynomial equations  $\{f = g \mid (f, g) \in T\}$  in  $B^n$ . In the following, we will call  $\{(f, g) \in \rho \mid (f, g) \in T\}$  a system of polynomial  $\rho$ -equations. A subset  $Y$  of  $B^n$  will be called an  $\rho$ -algebraic variety over  $A$  if there exists a non-empty subset  $T \subset S \times S$  such that  $Y = Z_\rho(T)(B)$ .

**Remark 4.2.** The motivation here comes from algebraic varieties (see [Ha, Chapter 1]). As we know, one of the main problem of algebraic geometry is to solve polynomial equations in rings. Let  $A$  and  $B$  be two commutative rings with  $A \subset B$ ,  $\{f_\alpha\}_{\alpha \in \Lambda} \subset A[x_1, \dots, x_n]$ , the solutions in  $B^n$  of the system of polynomial equations  $f_\alpha = 0$  ( $\alpha \in \Lambda$ ) is then the set  $Z(f_\alpha)_{\alpha \in \Lambda}(B) = \{P \in B^n : f_\alpha(P) = 0 \mid (\forall \alpha \in \Lambda)\}$  (see [Ha, p.2], [K, p.10]). By using the identity congruence  $\text{id}_B$  on  $B$ , this set can be rewritten as  $Z(f_\alpha)_{\alpha \in \Lambda}(B) = \{P \in B^n : (f_\alpha(P), 0) \in \text{id}_B \mid (\forall \alpha \in \Lambda)\}$ . So it is natural to consider the polynomial questions in semirings, in a sense of congruence, generalizing the identity congruence (i.e. equality), as stated in the above Def.4.1. Note that, in a ring, the equation  $f = g$  can be always changed to be  $f - g = 0$ , i.e., the form  $h = 0$ . The case for the system of polynomial equations  $(f_\alpha = g_\alpha)_{\alpha \in \Lambda}$  is similar. But, this is not true in a semiring, because usually there is no subtraction in a semiring. One can also consider the polynomial congruence equations or other related congruence equations in non-commutative semirings, yet it may be more complicated. Here we only consider the case of commutative semirings.



As an example, let  $A = \mathbb{Z}_{\geq 0}$  be the semiring as in Example 3.3 above. For each positive integer  $n$ , let  $\rho_n = \{(a, b) \in A \times A : a \equiv b \pmod{n}\}$ . Then obviously  $\rho_n$  is a congruence on  $A$ . For any polynomial  $f(x) \in \mathbb{Z}[x]$ , we have  $Z_{\rho_n}(f, 0)(A) = \{m \in A : (f(m), 0) \in \rho_n\} = \{m \in A : f(m) \equiv 0 \pmod{n}\} \rightarrow Z(f)(A/\rho_n) = Z(f)(\mathbb{Z}/n\mathbb{Z})$ , so  $\sharp Z_{\rho_n}(f, 0)(A) \geq \sharp Z(f)(\mathbb{Z}/n\mathbb{Z})$ .

**Lemma 4.3.** Let  $A, B, \rho$  and  $S = A[x_1, \dots, x_n]$  be as in Def.4.1 above.

Assume  $\rho \neq B \times B$ . Then we have

- (1)  $Z_{\rho}((0, 1))(B) = \emptyset$ , and  $Z_{\rho}((f, f))(B) = B^n$  ( $\forall f \in S$ ).
- (2) Let  $T_1$  and  $T_2$  be two non-empty subsets of  $S \times S$ , then  $Z_{\rho}(T_1)(B) \cup Z_{\rho}(T_2)(B) \subset Z_{\rho}(T_1 * T_2)(B)$ , where  $T_1 * T_2$  is defined as Def.2.3 above.
- (3) The intersection of any family of  $\rho$ -algebraic varieties is an  $\rho$ -algebraic variety.

**Proof.** (1) Obvious.

(2) Let  $P \in Z_{\rho}(T_1)(B)$ , then  $(f_1(P), g_1(P)) \in \rho$ , also  $(g_1(P), f_1(P)) \in \rho$  ( $\forall (f_1, g_1) \in T_1$ ). Now for any  $\alpha \in T_1 * T_2$ , by definition,  $\alpha = (f_1, g_1) * (f_2, g_2)$  for some  $(f_1, g_1) \in T_1$  and  $(f_2, g_2) \in T_2$ . So  $\alpha(P) = (f_1(P), g_1(P)) * (f_2(P), g_2(P)) = (f_1(P)f_2(P) + g_1(P)g_2(P), f_1(P)g_2(P) + f_2(P)g_1(P)) = (f_1(P)f_2(P) + g_1(P)g_2(P), f_2(P)(f_1(P), g_1(P)) + g_2(P)(g_1(P), f_1(P))) \in \rho$ , and so  $P \in Z_{\rho}(T_1 * T_2)(B)$ , which shows that  $Z_{\rho}(T_1)(B) \subset Z_{\rho}(T_1 * T_2)(B)$ . Similarly  $Z_{\rho}(T_2)(B) \subset Z_{\rho}(T_1 * T_2)(B)$ .

(3) If  $Y_{\alpha} = Z_{\rho}(T_{\alpha})(B)$  is any family of  $\rho$ -algebraic varieties, then  $\cap Y_{\alpha} = Z_{\rho}(\cup T_{\alpha})(B)$ , so  $\cap Y_{\alpha}$  is also an  $\rho$ -algebraic variety, and the proof is completed.  $\square$

**Theorem 4.4.** Let  $A, B$  and  $S = A[x_1, \dots, x_n]$  be as in Def.4.1 above. If  $\rho \in \text{Spec}^c(B)$  is a prime congruence on  $B$ , then for the  $\rho$ -algebraic varieties in  $B^n$  over  $A$ , we have

- (1) The union of two  $\rho$ –algebraic varieties is an  $\rho$ –algebraic variety;
- (2) The intersection of any family of  $\rho$ –algebraic varieties is an  $\rho$ –algebraic variety;
- (3) The empty set  $\emptyset$  and  $B^n$  are  $\rho$ –algebraic varieties.

**Proof.** (1) If  $Y_1 = Z_\rho(T_1)(B)$  and  $Y_2 = Z_\rho(T_2)(B)$  for some  $T_1, T_2 \subset S \times S$ , then by Lemma 4.3 above,  $Y_1 \cup Y_2 \subset Z_\rho(T_1 * T_2)(B)$ . Conversely, if  $P \in Z_\rho(T_1 * T_2)(B)$ , and  $P \notin Y_1$ , then there is an element  $(f_1, g_1) \in T_1$  such that  $(f_1(P), g_1(P)) \notin \rho$ . On the other hand, for any  $(f_2, g_2) \in T_2$ , we have  $(f_1, g_1) * (f_2, g_2) \in T_1 * T_2$ , so  $((f_1, g_1) * (f_2, g_2))(P) \in \rho$ , i.e.,  $(f_1(P), g_1(P)) * (f_2(P), g_2(P)) \in \rho$ , which implies that  $(f_2(P), g_2(P)) \in \rho$  since  $\rho$  is a prime congruence, so that  $P \in Y_2$ . Therefore  $Y_1 \cup Y_2 = Z_\rho(T_1 * T_2)(B)$  is an  $\rho$ –algebraic variety.

(2) and (3) follow from Lemma 4.3 above, and the proof is completed.  $\square$

**Definition 4.5.** Let  $A, B$  and  $S = A[x_1, \dots, x_n]$  be as in Def.4.1 above. If  $\rho \in \text{Spec}^c(B)$  is a prime congruence on  $B$ , then by Theorem 4.4 above, the set of all  $\rho$ –algebraic varieties in  $B^n$  over  $A$  satisfies the axiom of closed subsets, and give a topology  $\tau_{\rho, A}$  on  $B^n$ , i.e., a subset  $X$  of  $B^n$  is open in  $\tau_{\rho, A}$  if and only if its complement  $B^n \setminus X$  is an  $\rho$ –algebraic variety.  $\tau_{\rho, A}$  will be called the Zariski  $\rho$ –topology on  $B^n$ . For such  $\rho$ , a subset of  $B^n$  is equipped with the subspace topology induced from  $\tau_{\rho, A}$ . An  $\rho$ –algebraic variety  $X$  of  $B^n$  is irreducible if it can not be expressed as the union  $X = X_1 \cup X_2$  of two proper subsets, each one of which is closed in  $X$ .

**Definition 4.6.** Let  $A, B, \rho$  and  $S$  be as in Def.4.1 above. For any subset  $Y \subset B^n$ , we define

$$\rho_B(Y) = \{(f, g) \in S \times S : (f(P), g(P)) \in \rho \text{ for all } P \in Y\}.$$

Obviously,  $\rho_B(Y)$  is a congruence on  $S$ .

**Proposition 4.7.** Let  $A, B, \rho$  and  $S = A[x_1, \dots, x_n]$  be as in Def.4.1 above.

- (1) If  $T_1 \subset T_2$  are subsets of  $S \times S$ , then  $Z_\rho(T_1)(B) \supset Z_\rho(T_2)(B)$ .
- (2) If  $T \subset S \times S$ , then  $Z_\rho(T)(B) = Z_\rho(T^c)(B)$  and  $T \subset \rho_B(Z_\rho(T)(B))$ .
- (3) If  $Y_1 \subset Y_2$  are subsets of  $B^n$ , then  $\rho_B(Y_1) \supset \rho_B(Y_2)$ .
- (4) For any two subsets  $Y_1, Y_2$  of  $B^n$ , we have  $\rho_B(Y_1 \cup Y_2) = \rho_B(Y_1) \cap \rho_B(Y_2)$ .

**Proof.** (1) and (3) follow easily from the definitions.

(2)  $T \subset \rho_B(Z_\rho(T)(B))$  follows directly from the definition. For the equality, Since  $T \subset T^c$ , by (1),  $Z_\rho(T^c)(B) \subset Z_\rho(T)(B)$ . Conversely, let  $P \in Z_\rho(T)(B)$ , then  $(h_1(P), h_2(P)) \in \rho$  for all  $(h_1, h_2) \in T$ . For any  $(f, g) \in T^c$ , by Cor.2.17 above,  $f = g$  or for some positive integer  $n$  there exists a sequence  $f = f_1 \rightarrow f_2 \rightarrow \dots \rightarrow f_n = g$  in which for each  $i \in \{1, 2, \dots, n-1\}$  either  $(f_i, f_{i+1}) \in T^L$  or  $(f_{i+1}, f_i) \in T^L$ . If  $f = g$ , then obviously  $(f(P), g(P)) \in \rho$ . If  $(f_i, f_{i+1}) \in T^L$ , then  $(f_i, f_{i+1}) = (a_i g_i + h_i, b_i g_i + h_i)$  for some  $(a_i, b_i) \in T$  and some  $g_i, h_i \in S$ . By the choice of  $P$ ,  $(a_i(P), b_i(P)) \in \rho$ , so

$$\begin{aligned} (f_i(P), f_{i+1}(P)) &= (a_i(P)g_i(P) + h_i(P), b_i(P)g_i(P) + h_i(P)) \\ &= (a_i(P), b_i(P)) \cdot (g_i(P), g_i(P)) + (h_i(P), h_i(P)) \in \rho. \end{aligned}$$

Similarly, if  $(f_{i+1}, f_i) \in T^L$ , then  $(f_{i+1}(P), f_i(P)) \in \rho$ , which also implies that  $(f_i(P), f_{i+1}(P)) \in \rho$  because  $\rho$  is a congruence. So for the above sequence, we always have  $(f_i(P), f_{i+1}(P)) \in \rho$  for all  $i \in \{1, 2, \dots, n-1\}$ , that is,

$$(f_1(P), f_2(P)), (f_2(P), f_3(P)), \dots, (f_{n-1}(P), f_n(P)) \in \rho,$$

so  $(f_1(P), f_n(P))$ , i.e.,  $(f(P), g(P)) \in \rho$ . This shows that  $(f(P), g(P)) \in \rho$  for all

$(f, g) \in T^c$ , hence  $P \in Z_\rho(T^c)(B)$ , which implies that  $Z_\rho(T)(B) \subset Z_\rho(T^c)(B)$ .

Therefore,  $Z_\rho(T)(B) = Z_\rho(T^c)(B)$ .

(4) By (3),  $\rho_B(Y_1 \cup Y_2) \subset \rho_B(Y_i)$  ( $i = 1, 2$ ), so  $\rho_B(Y_1 \cup Y_2) \subset \rho_B(Y_1) \cap \rho_B(Y_2)$ . Conversely, if  $(f, g) \in \rho_B(Y_1) \cap \rho_B(Y_2)$ , then  $(f(P), g(P)) \in \rho$  for all  $P \in Y_i$  ( $i = 1, 2$ ), i.e.,  $(f(P), g(P)) \in \rho$  for all  $P \in Y_1 \cup Y_2$ , so  $(f, g) \in \rho_B(Y_1 \cup Y_2)$ , which implies that  $\rho_B(Y_1) \cap \rho_B(Y_2) \subset \rho_B(Y_1 \cup Y_2)$ . Therefore,  $\rho_B(Y_1 \cup Y_2) = \rho_B(Y_1) \cap \rho_B(Y_2)$ . The proof is completed.  $\square$

**Lemma 4.8.** Let  $A, B, \rho$  and  $S = A[x_1, \dots, x_n]$  be as in Def.4.1 above. For  $f, g \in S, P \in B^n$  and positive integer  $m$ , we have

$$(f, g)^{*m}(P) = (f(P), g(P))^{*m} \in B \times B \quad (\text{here we write } (f, g)(P) = (f(P), g(P))).$$

**Proof.** By Prop.2.6 above,

$$\begin{aligned} (f, g)^{*m} &= \left( \sum_{2|i, i=0}^m \binom{m}{i} f^{m-i} g^i, \sum_{2|i, i=1}^m \binom{m}{i} f^{m-i} g^i \right), \text{ so} \\ (f, g)^{*m}(P) &= \left( \sum_{2|i, i=0}^m \binom{m}{i} f(P)^{m-i} g(P)^i, \sum_{2|i, i=1}^m \binom{m}{i} f(P)^{m-i} g(P)^i \right) \\ &= (f(P), g(P))^{*m}. \end{aligned}$$

The proof is completed.  $\square$

**Proposition 4.9.** Let  $A, B, \rho$  and  $S = A[x_1, \dots, x_n]$  be as in Def.4.1 above.

Then for any congruence  $\sigma$  on  $S$ , we have  $\sqrt{\sigma} \subset (\sqrt{\rho})_B(Z_\rho(\sigma)(B))$ .

**Proof.** let  $(f, g) \in \sqrt{\sigma}$ , then  $(f + h, g + h)^{*m} \in \sigma$  for some  $h \in S$  and some positive integer  $m$ . For any  $P \in Z_\rho(\sigma)(B)$ ,  $(f_1(P), g_1(P)) \in \rho$  for all  $(f_1, g_1) \in \sigma$ , so in particular,  $(f + h, g + h)^{*m}(P) \in \rho$ . Then by Lemma 4.8 above,  $(f(P) + h(P), g(P) + h(P))^{*m} = (f + h, g + h)^{*m}(P) \in \rho$ , so  $(f(P), g(P)) \in \sqrt{\rho}$ , which implies that  $(f, g) \in (\sqrt{\rho})_B(Z_\rho(\sigma)(B))$ , and the proof is completed.  $\square$

**A question.** Under what conditions can the equality  $\sqrt{\sigma} = (\sqrt{\rho})_B(Z_\rho(\sigma)(B))$  hold, i.e., the Hilbert's Nullstellensatz hold ?

**Theorem 4.10.** Let  $A, B, \rho$  and  $S = A[x_1, \dots, x_n]$  be as in Def.4.1 above. If  $\rho \in \text{Spec}^c(B)$  is a prime congruence on  $B$ . Then for any subset  $Y \subset B^n$ , we have  $Z_\rho(\rho_B(Y))(B) = \overline{Y}$ , the closure of  $Y$  in  $B^n$  with the Zariski  $\rho$ -topology  $\tau_{\rho,A}$ .

**Proof.** Let  $P \in Y$ , then  $(f(P), g(P)) \in \rho$  for all  $(f, g) \in \rho_B(Y)$ , so  $P \in Z_\rho(\rho_B(Y))(B)$ . Thus  $Y \subset Z_\rho(\rho_B(Y))(B)$ . Since  $Z_\rho(\rho_B(Y))(B)$  is closed, we get  $\overline{Y} \subset Z_\rho(\rho_B(Y))(B)$ . On the other hand, let  $W$  be any closed subset containing  $Y$ . Then  $W = Z_\rho(\sigma)(B)$  for some congruence  $\sigma$  on  $S$ . So  $Z_\rho(\sigma)(B) \supset Y$ . By Prop.4.7 above,  $\sigma \subset \rho_B(Z_\rho(\sigma)(B)) \subset \rho_B(Y)$ . So  $W = Z_\rho(\sigma)(B) \supset Z_\rho(\rho_B(Y))(B)$ . Therefore,  $Z_\rho(\rho_B(Y))(B) = \overline{Y}$ , and the proof is completed.  $\square$

**Proposition 4.11.** Let  $A, B, \rho$  and  $S = A[x_1, \dots, x_n]$  be as in Theorem 4.10 above. Let  $Y \subset B^n$  be an  $\rho$ -algebraic variety. If  $Y$  is irreducible, then  $\rho_B(Y)$  is a prime congruence on  $S$ .

**Proof.** Let  $(f_1, g_1), (f_2, g_2) \in S \times S$ . If  $(f_1, g_1) * (f_2, g_2) \in \rho_B(Y)$ , then by Prop.4.7 and Theorem 4.10 above,  $Z_\rho((f_1, g_1) * (f_2, g_2))(B) \supset Z_\rho(\rho_B(Y))(B) = \overline{Y} = Y$ . From the proof of Theorem 4.4(1) above, we have  $Z_\rho((f_1, g_1) * (f_2, g_2))(B) = Z_\rho((f_1, g_1))(B) \cup Z_\rho((f_2, g_2))(B)$ , so  $Y \subset Z_\rho((f_1, g_1))(B) \cup Z_\rho((f_2, g_2))(B)$ . Thus  $Y = (Y \cap Z_\rho((f_1, g_1))(B)) \cup (Y \cap Z_\rho((f_2, g_2))(B))$ , both being closed subsets of  $Y$ . Since  $Y$  is irreducible, we have either  $Y = Y \cap Z_\rho((f_1, g_1))(B)$ , in which case  $Y \subset Z_\rho((f_1, g_1))(B)$ , or  $Y \subset Z_\rho((f_2, g_2))(B)$ . So by Prop.4.7 above,  $(f_1, g_1) \in \rho_B(Z_\rho((f_1, g_1))(B)) \subset \rho_B(Y)$ , i.e.,  $(f_1, g_1) \in \rho_B(Y)$ , or  $(f_2, g_2) \in \rho_B(Y)$ , and so

$\rho_B(Y)$  is a prime congruence. The proof is completed.  $\square$

Let  $A, B$  be two commutative semirings, and  $\phi$  be a homomorphism from  $A$  to  $B$ , i.e.,  $\phi : A \rightarrow B$  is a map such that  $\phi(0) = 0, \phi(1) = 1$ , and for all  $a, b \in A$ ,  $\phi(a+b) = \phi(a) + \phi(b)$  and  $\phi(ab) = \phi(a) \cdot \phi(b)$ . Then the kernel  $\ker\phi = \{(a_1, a_2) \in A \times A : \phi(a_1) = \phi(a_2)\}$  is a congruence on  $A$ , and  $\phi$  induces a unique injective homomorphism, say  $\bar{\phi} : A/\ker\phi \rightarrow B$  such that  $\phi = \bar{\phi} \circ \eta$ , where  $\eta : A \rightarrow A/\ker\phi$  is the natural surjective homomorphism. Moreover, via such  $\phi$ ,  $B$  is an  $A$ -semimodule, hence an  $A$ -semialgebra. In general, for a commutative semiring  $A$ , a set  $B$  is an  $A$ -semialgebra if  $B$  is both a commutative semiring and an  $A$ -semimodule such that  $a(bc) = (ab)c = b(ac)$  ( $\forall a \in A, b, c \in B$ ). For two  $A$ -semialgebras  $B$  and  $C$ , a map  $\phi : B \rightarrow C$  is an  $A$ -semialgebra homomorphism if  $\phi$  is both a semiring homomorphism and an  $A$ -semimodule homomorphism. We let  $\text{Hom}_{A\text{-alg}}(B, C)$  to denote the set of all  $A$ -semialgebra homomorphisms from  $B$  to  $C$ .

Now come back to our semirings  $A \subset B$  and  $S = A[x_1, \dots, x_n]$  as in Def.4.1 above. Let  $\rho$  be a congruence on  $B$ , and  $T \subset S \times S$  be a non-empty subset. Recall that  $T^c$  is the congruence on  $S$  generated by  $T$ . Then it is easy to see that both the quotient semirings  $S/T^c$  and  $B/\rho$  are  $A$ -semialgebras. Recall that for a subset  $Y$  of  $B^n$ ,  $Y/\rho = \{(\bar{c}_1, \dots, \bar{c}_n) : (c_1, \dots, c_n) \in Y\} \subset B^n/\rho$  with  $\bar{c}_i = c_i \bmod \rho \in B/\rho$ .

**Theorem 4.12.** Let  $A, B$  and  $S = A[x_1, \dots, x_n]$  be as in Def.4.1 above.

Let  $\rho$  be a congruence on  $B$ , and  $T \subset S \times S$  be a non-empty subset. Then there exists an one-to-one map of  $Z_\rho(T)(B)/\rho$  onto  $\text{Hom}_{A\text{-alg}}(S/T^c, B/\rho)$ , so the cardinals  $\sharp Z_\rho(T)(B)/\rho = \sharp \text{Hom}_{A\text{-alg}}(S/T^c, B/\rho)$ . In particular, if  $\rho = \text{id}_B$ , then  $\sharp Z_{\text{id}_B}(T)(B) = \sharp \text{Hom}_{A\text{-alg}}(S/T^c, B)$ .

**Proof.** By the composition of homomorphisms  $A \hookrightarrow B \rightarrow B/\rho$  (resp.  $A \hookrightarrow S \rightarrow S/T^c$ ),  $B/\rho$  (resp.  $S/T^c$ ) becomes a natural  $A$ -semialgebra. Define a map

$$\phi_0 : Z_\rho(T)(B) \longrightarrow \text{Hom}_{A\text{-alg}}(S/T^c, B/\rho), \quad P \mapsto \phi_0(P),$$

where  $\phi_0(P) : S/T^c \longrightarrow B/\rho$  is defined as follows:

Write  $P = (b_1, \dots, b_n)$  with  $b_i \in B$  ( $i = 1, \dots, n$ ), we have a homomorphism of semirings

$$\gamma_P : S \longrightarrow B/\rho, \quad h(x_1, \dots, x_n) \mapsto \overline{h(b_1, \dots, b_n)} = \overline{h(P)} \quad (\forall h \in S).$$

For any  $(f, g) \in T$ ,  $(f(P), g(P)) \in \rho$  since  $P \in Z_\rho(T)(B)$ . So  $\gamma_P(f) = \overline{f(P)} = \overline{g(P)} = \gamma_P(g)$ , i.e.,  $(f, g) \in \ker \gamma_P$ , so  $T \subset \ker \gamma_P$ , and so  $T^c \subset \ker \gamma_P$ . Therefore,

there is a unique homomorphism of semirings, say  $\phi_0(P) : S/T^c \longrightarrow B/\rho$  such that

$$\phi_0(P) \circ \eta = \gamma_P, \quad \text{where } \eta : S \longrightarrow S/T^c \text{ is the natural surjective homomorphism.}$$

Moreover, for any  $a \in A$  and  $f \in S$ ,  $\phi_0(P)(a \cdot \overline{f}) = \phi_0(P)(\overline{af}) = \gamma_P(af) = \overline{(af)(P)} = \overline{a \cdot f(P)} = \overline{a} \cdot \overline{f(P)} = \overline{a} \cdot \gamma_P(f) = \overline{a} \cdot \phi_0(P)(\overline{f})$ , so  $\phi_0(P)$  is also an

$A$ -semimodule homomorphism, hence  $\phi_0(P) \in \text{Hom}_{A\text{-alg}}(S/T^c, B/\rho)$ . By this way,

the map  $\phi_0$  is given. Now let  $P, Q \in Z_\rho(T)(B)$  be two points such that  $\phi_0(P) =$

$$\phi_0(Q), \text{ then for any } f \in S, \gamma_P(f) = \phi_0(P)(\eta(f)) = \phi_0(P)(\overline{f}) = \phi_0(Q)(\overline{f}) = \gamma_Q(f).$$

So  $\gamma_P = \gamma_Q$ . If we write  $P = (b_1, \dots, b_n)$  and  $Q = (c_1, \dots, c_n)$ , then coordinates

$$b_i = x_i(P) \text{ and } c_i = x_i(Q). \text{ So } \overline{b_i} = \overline{x_i(P)} = \gamma_P(x_i) = \gamma_Q(x_i) = \overline{x_i(Q)} = \overline{c_i}, \text{ and}$$

so  $\overline{P} = (\overline{b_1}, \dots, \overline{b_n}) = (\overline{c_1}, \dots, \overline{c_n}) = \overline{Q}$ , i.e.,  $\overline{P} = \overline{Q} \in Z_\rho(T)(B)/\rho$ . Conversely, if

$P, Q \in Z_\rho(T)(B)$  satisfy  $\overline{P} = \overline{Q} \in Z_\rho(T)(B)/\rho$ , then obviously,  $\gamma_P = \gamma_Q$ , and so

$$\phi_0(P) = \phi_0(Q). \text{ Therefore, } \phi_0(P) = \phi_0(Q) \Leftrightarrow \overline{P} = \overline{Q}, \text{ and so } \phi_0 \text{ induces an injective}$$

map

$$\phi : Z_\rho(T)(B)/\rho \longrightarrow \text{Hom}_{A\text{-alg}}(S/T^c, B/\rho), \quad \overline{P} \mapsto \phi_0(P) \quad (\forall P \in Z_\rho(T)(B)).$$

Next, we define a map

$$\psi : \text{Hom}_{A\text{-alg}}(S/T^c, B/\rho) \longrightarrow Z_\rho(T)(B)/\rho, \quad \gamma \mapsto \psi(\gamma),$$

where the point  $\psi(\gamma) \in Z_\rho(T)(B)/\rho$  is defined as follows:

By composing  $\gamma$  with the natural  $A$ -semialgebra homomorphism  $\eta : S \longrightarrow S/T^c$ , we get an  $A$ -semialgebra homomorphism  $\beta = \gamma \circ \eta : S \longrightarrow B/\rho$ . For each  $i = 1, \dots, n$  write  $u_i = \beta(x_i) = \gamma(\eta(x_i)) \in B/\rho$ , so  $u_i = \overline{b_i}$  for some  $b_i \in B$ . Then we define  $\psi(\gamma) = (\overline{b_1}, \dots, \overline{b_n}) \in B^n/\rho$ . We need to show that  $(b_1, \dots, b_n) \in Z_\rho(T)(B)/\rho$ . For this, write  $P = (b_1, \dots, b_n) \in B^n$ . For any  $(f, g) \in T(\subset T^c)$ ,  $\overline{f} = \overline{g} \in S/T^c$ , so  $\beta(f) = \gamma(\eta(f)) = \gamma(\overline{f}) = \gamma(\overline{g}) = \gamma(\eta(g)) = \beta(g)$ . Note that  $f = f(x_1, \dots, x_n)$ ,  $g = g(x_1, \dots, x_n)$  and  $\beta$  is an  $A$ -semialgebra homomorphism, we have

$$\begin{aligned} f(\beta(x_1), \dots, \beta(x_n)) &= \beta(f(x_1, \dots, x_n)) = \beta(f) \\ &= \beta(g) = \beta(g(x_1, \dots, x_n)) = g(\beta(x_1), \dots, \beta(x_n)), \end{aligned}$$

i.e.,  $f(\overline{b_1}, \dots, \overline{b_n}) = g(\overline{b_1}, \dots, \overline{b_n}) \in B/\rho$ , which means  $\overline{f(P)} = \overline{g(P)}$ , so  $(f(P), g(P)) \in \rho$ , which implies that  $P \in Z_\rho(T)(B)$ . So the above  $\psi(\gamma) = \overline{P} \in Z_\rho(T)(B)/\rho$ . By this way, the map  $\psi$  is given.

Now we consider the composition map

$$\psi \circ \phi : Z_\rho(T)(B)/\rho \longrightarrow \text{Hom}_{A\text{-alg}}(S/T^c, B/\rho) \longrightarrow Z_\rho(T)(B)/\rho.$$

For any  $P \in Z_\rho(T)(B)$ , by definition,

$$\begin{aligned} (\psi \circ \phi)(\overline{P}) &= \psi(\phi(\overline{P})) = \psi(\phi_0(P)) = (\phi_0(P)(\overline{x_1}), \dots, \phi_0(P)(\overline{x_n})) \\ &= (\gamma_P(x_1), \dots, \gamma_P(x_n)) = (\overline{x_1(P)}, \dots, \overline{x_n(P)}) = \overline{P}. \end{aligned}$$

So  $\psi \circ \phi = \text{id}$  is the identity map.

Also for the composition map



$$\phi \circ \psi : \text{Hom}_{A\text{-alg}}(S/T^c, B/\rho) \longrightarrow Z_\rho(T)(B)/\rho \longrightarrow \text{Hom}_{A\text{-alg}}(S/T^c, B/\rho).$$

For any  $\gamma \in \text{Hom}_{A\text{-alg}}(S/T^c, B/\rho)$ , write  $\psi(\gamma) = \overline{P}$  for some  $P = (b_1, \dots, b_n) \in Z_\rho(T)(B)$ . Then by definition,  $\psi(\gamma) = (\gamma(\overline{x_1}), \dots, \gamma(\overline{x_n}))$ , so  $\gamma(\overline{x_i}) = \overline{b_i}$  ( $i = 1, \dots, n$ ). On the other hand, by definition,  $\phi_0(P)(\overline{x_i}) = \gamma_P(x_i) = \overline{x_i(P)} = \overline{b_i}$  ( $i = 1, \dots, n$ ). So  $\phi_0(P)(\overline{x_i}) = \gamma(\overline{x_i})$ , and hence  $\phi_0(P) = \gamma$ , i.e.,  $(\phi \circ \psi)(\gamma) = \phi(\overline{P}) = \phi_0(P) = \gamma$ , which implies that  $\phi \circ \psi = \text{id}$  is the identity map. Therefore, both  $\phi$  and  $\psi$  are bijective, so  $\sharp Z_\rho(T)(B)/\rho = \sharp \text{Hom}_{A\text{-alg}}(S/T^c, B/\rho)$ , and the proof is completed.  $\square$

**Example 4.13.** (1) Let  $A = B = \mathbb{C}$ , the field of complex numbers, and  $S = \mathbb{C}[x_1, \dots, x_n]$ , the ring of polynomials over  $\mathbb{C}$ . Let  $\emptyset \neq T \subset S \times S$ , and denote  $T_0 = \{f - g : (f, g) \in T\}$ . Take  $\rho = \text{id}_B = \text{id}_{\mathbb{C}}$ . Then Obviously,  $Z_\rho(T)(B)/\rho = Z_{\text{id}}(T)(\mathbb{C}) = Z(T_0)$  (the common complex zeros of all  $f \in T_0$ ). Also for the congruence  $T^c$  generated by  $T$ , it is easy to see that the ideal  $I_{T^c} = \{f \in S : (f, 0) \in T^c\} = \langle T_0 \rangle$ , where  $\langle T_0 \rangle$  is the ideal of  $S$  generated by  $T_0$ . So  $S/T^c = S/\langle T_0 \rangle$ . Then by Theorem 4.12 above, there is a bijective map between  $Z(T_0) = \sharp Z_\rho(T)(B)/\rho$  and  $\text{Hom}_{\mathbb{C}\text{-alg}}(S/\langle T_0 \rangle, \mathbb{C})$ , which shows the same meaning for the algebraic variety in [Ha].

(2) Let  $A = B = \mathbb{Z}$ , the ring of integers, and  $S = \mathbb{Z}[x_1, \dots, x_n]$ , the ring of polynomials over  $\mathbb{Z}$ . Let  $\emptyset \neq T \subset S \times S$ , and denote  $T_0 = \{f - g : (f, g) \in T\}$ . Take a prime number  $p$  and let  $\rho$  be the modulo  $p$  congruence on  $B$ , i.e.,  $\rho = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a \equiv b \pmod{p}\}$ . Then it is easy to see that  $S/T^c = S/\langle T_0 \rangle$ . Note that  $B/\rho = \mathbb{F}_p$ , the field of  $p$  elements. So  $\text{Hom}_{A\text{-alg}}(S/T^c, B/\rho) = \text{Hom}_{\mathbb{Z}\text{-alg}}(S/\langle T_0 \rangle, \mathbb{F}_p)$ . On

the other hand,

$$\begin{aligned}
Z_\rho(T)(B)/\rho &= \{P \in \mathbb{Z}^n : (f(P), g(P)) \in \rho \ (\forall (f, g) \in T)\} \bmod p \\
&= \{P \in \mathbb{Z}^n : f(P) \equiv g(P) \bmod p \ (\forall (f, g) \in T)\} \bmod p \\
&= \{P \in \mathbb{Z}^n : f(P) \equiv 0 \bmod p \ (\forall f \in T_0)\} \bmod p \\
&= \{P \in \mathbb{F}_p^n : f(P) = \bar{0} \ (\forall f \in T_0)\} = Z(T_0)(\mathbb{F}_p),
\end{aligned}$$

the zero set of  $\mathbb{F}_p$ -rational points of the system of polynomial equations  $(f = 0)_{f \in T_0}$ , and  $\sharp Z(T_0)(\mathbb{F}_p) = \sharp \text{Hom}_{\mathbb{Z}\text{-alg}}(S / \langle T_0 \rangle, \mathbb{F}_p)$ , the same as shown in Theorem 4.12 above.  $\square$

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